WHITNEY UMBRELLAS AND SWALLOWTAILS

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Dedicated to Professor Shyuichi Izumiya on the occasion of his sixtieth birthday

Abstract. In this paper, we introduce the notions of map-germs of pedal unfolding type and normalized Legendrian map-germs; and then we show that the fundamental theorem of calculus provides a natural one to one correspondence between Whitney umbrellas of pedal unfolding type and normalized swallowtails.

1. Introduction

The map-germ

$$f(x,y) = (xy, x^2, y) \tag{1}$$

is well-known as the normal form of Whitney umbrella after Whitney's pioneer works [10, 11]. For the map-germ (1), compose the following two coordinate transformations: $h_s(x,y) = (x, x^2+y)$ and $h_t(X,Y,Z) = (X, -Z, -Y+Z)$ where (X,Y,Z) is the standard coordinates of the target space $\mathbb{R}^3$. Then, we have the following:

$$g(x,y) = h_t \circ f \circ h_s(x,y) = (x^3+xy, -x^2-y, y). \tag{2}$$

Put

$$G(x,y) = \left(\int_0^x (x^3+xy)dx, \int_0^x (-x^2-y)dx, y\right) = \left(\frac{1}{4}x^4 + \frac{1}{2}x^2 y, -\frac{1}{3}x^3 - xy, y\right). \tag{3}$$

For the map-germ (3), compose the following two scaling transformations: $H_s(x,y) = (x, \frac{1}{6}y)$ and $H_t(X,Y,Z) = (12X, 12Y, 6Z)$. Then, we have the following:

$$H_t \circ G \circ H_s(x,y) = (3x^4 + x^2 y, -4x^3 - 2xy, y). \tag{4}$$

The map-germ (4) is well-known as the normal form of swallowtail (for instance, see [2] p.129).

Two C^∞ map-germs $\varphi, \psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are said to be $\mathcal{A}$-equivalent if there exist germs of C^∞ diffeomorphisms $h_s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $h_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\psi = h_t \circ \varphi \circ h_s$. A C^∞ map-germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called a *Whitney umbrella* (resp., *swallowtail*) if φ is $\mathcal{A}$-equivalent to (1) (resp., (4)). As above, the Whitney umbrella (1) produces the swallowtail (4) via (2) and (3). By the converse procedure, the swallowtail (4) produces the Whitney umbrella (1).

Note that it is impossible to produce a swallowtail by integrating (1) directly. This is because the discriminant set of (4) is not diffeomorphic to the discriminant

2010 *Mathematics Subject Classification.* 57R45, 58K25, 53A05.

Key words and phrases. Whitney umbrella of pedal unfolding type, normalized swallowtail.

set of the following (5):

$$(x,y)\mapsto\left(\int_0^x xydx,\int_0^x x^2dx,y\right). \tag{5}$$

Note further that the form (2) may be written as follows:

$$g(x,y)=\left(x(x^2+y),-(x^2+y),y\right)=\left(b\left(-x,-(x^2+y)\right),y\right),$$

where $b(X,Y)=(XY,Y)$ (b stands for "the blow down").

Definition 1. (i) A C^∞ map-germ $\varphi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ having the following form (6) is said to be *of pedal unfolding type*.

$$\varphi(x,y)=(n(x,y)p(x,y),p(x,y),y)=(b\left(n(x,y),p(x,y)\right),y) \tag{6}$$

where $n:(\mathbb{R}^2,0)\to(\mathbb{R},0)$ is a C^∞ function-germ such that $\frac{\partial n}{\partial x}(0,0)\neq 0$ and $p:(\mathbb{R}^2,0)\to(\mathbb{R},0)$ is a C^∞ function-germ.

(ii) For a C^∞ map-germ of pedal unfolding type $\varphi(x,y)=(n(x,y)p(x,y),p(x,y),y)$, put

$$\mathcal{I}(\varphi)=\left(\int_0^x n(x,y)p(x,y)dx,\int_0^x p(x,y)dx,y\right).$$

The map-germ $\mathcal{I}(\varphi):(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ is called the *integration of* φ.

(iii) A C^∞ map-germ $\Phi:(\mathbb{R}^m,0)\to(\mathbb{R}^{m+1},0)$ is called a *Legendrian map-germ* if there exists a germ of C^∞ vector field $\nu_\Phi:(\mathbb{R}^m,0)\to T_1\mathbb{R}^{m+1}$ along Φ such that the following conditions hold where the dot in the center stands for the scalar product of two vectors of $T_{\Phi(x,y)}\mathbb{R}^{m+1}$ and $T_1\mathbb{R}^{m+1}$ stands for the unit tangent bundle of $\mathbb{R}^{m+1}$.

(a) $\frac{\partial\Phi}{\partial x_1}(x_1,\ldots,x_m)\cdot\nu_\Phi(x_1,\ldots,x_m)=\cdots=\frac{\partial\Phi}{\partial x_m}(x_1,\ldots,x_m)\cdot\nu_\Phi(x_1,\ldots,x_m)=0$.

(b) The map-germ $L_\Phi:(\mathbb{R}^m,0)\to T_1\mathbb{R}^{m+1}$ defined by

$$L_\Phi(x_1,\ldots,x_m)=(\Phi(x_1,\ldots,x_m),\nu_\Phi(x_1,\ldots,x_m))$$

is non-singular. The map-germ L_Φ is called a *Legendrian lift of* Φ.

The C^∞ vector field ν_Φ is called a *unit normal vector field of* Φ.

(iv) A Legendrian map-germ $\Phi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ is said to be *normalized* if Φ satisfies the following three conditions:

(a) Φ has the form $\Phi(x,y)=(\Phi_1(x,y),\Phi_2(x,y),y)$.

(b) $\frac{\partial\Phi_2}{\partial x}(0,0)=0$.

(c) $\nu_\Phi(0,0)$ is $\frac{\partial}{\partial X}$ or $-\frac{\partial}{\partial X}$.

(v) For a normalized Legendrian map-germ $\Phi(x,y)=(\Phi_1(x,y),\Phi_2(x,y),y)$, put

$$\mathcal{D}(\Phi)=\left(\frac{\partial\Phi_1}{\partial x}(x,y),\frac{\partial\Phi_2}{\partial x}(x,y),y\right).$$

The map-germ $\mathcal{D}(\Phi):(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ is called the *differentiation of* Φ.

Since any map-germ $\varphi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ which is a germ of one-parameter pedal unfolding of a spherical pedal curve has the form (6) (see [7]), it is reasonable that a map-germ φ having the form (6) is said to be of pedal unfolding type. As shown in [7], not only non-singular map-germs but also Whitney umbrellas may be realized as germs of one-parameter pedal unfoldings of spherical pedal curves. For details on Legendrian map-germs, see [1, 3, 12, 13]. Note that both (3) and (7) are normalized Legendrian map-germs.

Proposition 1. (i) *For a C^∞ map-germ of pedal unfolding type $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, $\mathcal{I}(\varphi)$ is a normalized Legendrian map-germ.*
(ii) *For a normalized Legendrian map-germ $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, $\mathcal{D}(\Phi)$ is a map-germ of pedal unfolding type.*

Put

$$\begin{aligned} \mathcal{W} &= \left\{\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{ Whitney umbrella of pedal unfolding type}\right\}, \\ \mathcal{S} &= \left\{\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{ normalized swallowtail}\right\}. \end{aligned}$$

The main purpose of this paper is to show the following Theorem 1:

Theorem 1. (i) *The map $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ defined by $\mathcal{W} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \mathcal{S}$ is well-defined and bijective.*
(ii) *The map $\mathcal{D} : \mathcal{S} \to \mathcal{W}$ defined by $\mathcal{S} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{W}$ is well-defined and bijective.*

Incidentally, we show the following Theorem 2. A C^∞ map-germ $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called a *cuspidal edge* if Φ is $\mathcal{A}$-equivalent to the following (7) :

$$(x, y) \mapsto \left(\frac{1}{3}x^3, \frac{1}{2}x^2, y\right). \tag{7}$$

Put

$$\begin{aligned} \mathcal{N} &= \left\{\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{ non-singular map-germ of pedal unfolding type}\right\}, \\ \mathcal{C} &= \left\{\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{ normalized cuspidal edge}\right\}. \end{aligned}$$

Theorem 2. (i) *The map $\mathcal{I} : \mathcal{N} \to \mathcal{C}$ defined by $\mathcal{N} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \mathcal{C}$ is well-defined and bijective.*
(ii) *The map $\mathcal{D} : \mathcal{C} \to \mathcal{N}$ defined by $\mathcal{C} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{N}$ is well-defined and bijective.*

Both the following two are well-known (for instance, see [1]).

(i) Any stable map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is either a Whitney umbrella or non-singular.
(ii) Any Legendrian stable singularity $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is either a swallowtail or a cuspidal edge.

Therefore, Theorems 1 and 2 may be regarded as the fundamental theorem of calculus for stable map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ and Legendrian stable singularities $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$.

Theorems 1 and 2 yield the following conjecture naturally.

Conjecture 1. (i) *Let $\varphi_1, \varphi_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be two C^∞ map-germs of pedal unfolding type. Suppose that φ_1 is $\mathcal{A}$-equivalent to φ_2. Then, $\mathcal{I}(\varphi_1)$ is $\mathcal{A}$-equivalent to $\mathcal{I}(\varphi_2)$.*
(ii) *Let $\Phi_1, \Phi_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be two normalized Legendrian map-germs. Suppose that Φ_1 is $\mathcal{A}$-equivalent to Φ_2. Then, $\mathcal{D}(\Phi_1)$ is $\mathcal{A}$-equivalent to $\mathcal{D}(\Phi_2)$.*

In §2, several preparations for the proofs of Theorems 1 and 2 and the proof of Proposition 1 are given. Theorems 1 and 2 are proved in §3 and §4 respectively.

2. Preliminaries

2.1. **Function-germs with two variables and map-germs with two variables.** Let $\mathcal{E}_2$ be the set of C^∞ function-germs $(\mathbb{R}^2, 0) \to \mathbb{R}$ and let m_2 be the subset of $\mathcal{E}_2$ consisting of C^∞ function-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. The sets $\mathcal{E}_2$ have natural $\mathbb{R}$-algebra structures. For a C^∞ map-germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, let $\varphi^* : \mathcal{E}_2 \to \mathcal{E}_2$ be the $\mathbb{R}$-algebra homomorphism defined by $\varphi^*(u) = u \circ \varphi$. Put $Q(\varphi) = \mathcal{E}_2/\varphi^* m_2 \mathcal{E}_2$. Then, $Q(\varphi)$ is an $\mathbb{R}$-algebra. The following Proposition 2 is a special case of theorem (2.1) of [5].

Proposition 2. *Let $p : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a C^∞ function-germ. Then, the following hold:*

(i) *The $\mathbb{R}$-algebra $Q(p(x, y), y)$ is isomorphic to $Q(x^2, y)$ if and only if $\frac{\partial p}{\partial x}(0, 0) = 0$ and $\frac{\partial^2 p}{\partial x^2}(0, 0) \neq 0$.*

(ii) *The $\mathbb{R}$-algebra $Q(p(x, y), y)$ is isomorphic to $Q(x, y)$ if and only if $(x, y) \mapsto (p(x, y), y)$ is a germ of C^∞ diffeomorphism.*

Definition 2 ([6])**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the form $T(s, \lambda) = (-s, \lambda)$. Two C^∞ function-germs $p_1, p_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are said to be *$\mathcal{K}^T$-equivalent* if there exist a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of the form $h \circ T = T \circ h$ and a C^∞ function-germ $M : (\mathbb{R}^2, (0, 0)) \to \mathbb{R} - \{0\}$ of the form $M \circ T = M$ such that $p_1 \circ h(x, y) = M(x, y)p_2(x, y)$.

Theorem 3 ([6])**.** *Two C^∞ map-germs $\varphi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $(i = 1, 2)$ of the following form*

$$\varphi_i(x, y) = (xp_i(x^2, y), x^2, y) \quad \textit{where } p_i(x^2, y) \notin m_2^\infty, \quad (i = 1,\ 2)$$

are $\mathcal{A}$-equivalent if and only if the function-germs $p_i(x^2, y)$ are $\mathcal{K}^T$-equivalent.

Here, $m_2^\infty = \{q : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \mid \frac{\partial^{i+j} q}{\partial x^i \partial y^j}(0, 0) = 0 \ (\forall i, j \in \{0\} \cup \mathbb{N})\}$. By Theorem 3 and the Malgrange preparation theorem (for instance, see [1]), the following holds:

Corollary 1. *Two C^∞ map-germs $\varphi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $(i = 1, 2)$ of the following form*

$$\varphi_i(x, y) = (n_i(x, y)p_i(x^2, y), x^2, y),$$

where $p_i(x^2, y) \notin m_2^\infty$ and $n_i(x, y)$ satisfies $\frac{\partial n_i}{\partial x}(0, 0) \neq 0$ for each $i \in \{1, 2\}$, are $\mathcal{A}$-equivalent if and only if the function-germs $p_i(x^2, y)$ are $\mathcal{K}^T$-equivalent.

2.2. **Map-germs of pedal unfolding type.** Let $\varphi : I \times J \to \mathbb{R}^3$ be a representative of a given C^∞ map-germ of pedal unfolding type, where I, J be a sufficiently small intervals containing the origin of $\mathbb{R}$. Then, we may put $\varphi(x, y) = (n(x, y)p(x, y), p(x, y))$. Put

$$\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y) = \left(\int_0^x n(x, y)p(x, y)dx, \int_0^x p(x, y)dx, y \right)$$

and

$$\widetilde{\mu}_\Phi(x, y) = \frac{\partial}{\partial X} - n(x, y)\frac{\partial}{\partial Y}.$$

Since $\widetilde{\mu}_\Phi(x, y) \neq 0$ for any $x \in I$ and $y \in J$, for any fixed $y \in J$ we may define the map-germ $L_{\Phi,y} : (\mathbb{R}, 0) \to T_1\mathbb{R}^2$ as

$$L_{\Phi,y}(x) = \left((\Phi_1(x, y), \Phi_2(x, y)), \frac{\widetilde{\mu}_\Phi(x, y)}{||\widetilde{\mu}_\Phi(x, y)||} \right),$$

where $T_1\mathbb{R}^2$ is the unit tangent bundle of $\mathbb{R}^2$. Then, since φ is a representative of a map-germ of pedal unfolding type, we have the following:

Lemma 2.1. *For any $y \in J$, $L_{\Phi,y} : (\mathbb{R},0) \to T_1\mathbb{R}^2$ is a Legendrian lift of the map-germ $x \mapsto (\Phi_1(x,y), \Phi_2(x,y))$.*

By Lemma 2.1, we have the following:

Lemma 2.2. *For any $y \in J$, the map-germ $\widetilde{\Phi}_y : (\mathbb{R},0) \to (\mathbb{R}^2,0)$ defined by $\widetilde{\Phi}_y(x) = (\Phi_1(x,y), \Phi_2(x,y))$ is a Legendrian map-germ.*

Next, put

$$\widetilde{\nu}_\Phi(x,y) = \widetilde{\mu}_\Phi(x,y) - \left(\frac{\partial \Phi_1}{\partial y}(x,y) - n(x,y)\frac{\partial \Phi_2}{\partial y}(x,y)\right)\frac{\partial}{\partial Z}.$$

Then, we have the following:

Lemma 2.3. *For any $x \in I$ and $y \in J$,*

$$\widetilde{\nu}_\Phi(x,y) \cdot \frac{\partial \Phi}{\partial x}(x,y) = 0, \quad \widetilde{\nu}_\Phi(x,y) \cdot \frac{\partial \Phi}{\partial y}(x,y) = 0.$$

Since $\widetilde{\nu}_\Phi(x,y) \neq 0$ for any $x \in I$ and $y \in J$, we may define the map-germ $L_\Phi : (\mathbb{R}^2,0) \to T_1\mathbb{R}^3$ as

$$L_\Phi(x,y) = \left(\Phi(x,y), \frac{\widetilde{\nu}_\Phi(x,y)}{||\widetilde{\nu}_\Phi(x,y)||}\right).$$

Then, by Lemma 2.3 we have the following:

Lemma 2.4. *$L_\Phi : (\mathbb{R}^2,0) \to T_1\mathbb{R}^3$ is a Legendrian lift of Φ.*

By Lemma 2.4 we have the following:

Lemma 2.5. *$\Phi : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ is a Legendrian map-germ.*

2.3. **Normalized Legendrian map-germs.** Let $\Phi : U \to \mathbb{R}^3$ be a representative of a given normalized Legendrian map-germ $(\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$, where U is a sufficiently small neighborhood of the origin of $\mathbb{R}^2$. We assume that the origin of $\mathbb{R}^2$ is a singular point of Φ. By the condition (a) of the definition of normalized Legendrian map-germs, we may assume that Φ has the form $\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y), y)$. Since Φ is a a representative of a Legendrian map-germ, we have the following:

Lemma 2.6. *There exists a C^∞ vector field ν_Φ along Φ,*

$$\nu_\Phi(x,y) = n_1(x,y)\frac{\partial}{\partial X} + n_2(x,y)\frac{\partial}{\partial Y} + n_3(x,y)\frac{\partial}{\partial Z},$$

such that the following three hold:

(i) $n_1(x,y)\frac{\partial \Phi_1}{\partial x}(x,y) + n_2(x,y)\frac{\partial \Phi_2}{\partial x}(x,y) = 0$.
(ii) $n_1(x,y)\frac{\partial \Phi_1}{\partial y}(x,y) + n_2(x,y)\frac{\partial \Phi_2}{\partial y}(x,y) + n_3(x,y) = 0$.
(iii) *The map $L_\Phi : U \to T_1\mathbb{R}^3$ defined by $L_\Phi(x,y) = (\Phi(x,y), \nu_\Phi(x,y))$ is an immersion.*

By the condition (c) of the definition of normalized Legendrian map-germs, we have the following:

Lemma 2.7. *For the vector field ν_Φ, $n_1(0,0) \neq 0$ and $n_2(0,0) = n_3(0,0) = 0$.*

By the assertion (i) of Lemma 2.6 and Lemma 2.7, we have the following equality as function-germs.

$$\frac{\partial \Phi_1}{\partial x}(x,y) = -\frac{n_2(x,y)}{n_1(x,y)}\frac{\partial \Phi_2}{\partial x}(x,y). \tag{8}$$

Then, by the condition (b) of the definition of normalized Legendrian maps and the equality (8), the following holds:

Lemma 2.8. *The map-germ $\mathcal{D}(\Phi)$ maps the origin to the origin.*

Put

$$n(x,y) = -\frac{n_2(x,y)}{n_1(x,y)} \text{ and } p(x,y) = \frac{\partial \Phi_2}{\partial x}(x,y).$$

Then, we have clearly the following:

Lemma 2.9. *Both function-germs n and p are of class C^∞ and the equality $\mathcal{D}(\Phi)(x,y) = (n(x,y)p(x,y), p(x,y), y)$ holds.*

Furthrmore, we have the following:

Lemma 2.10. *The function-germ n satisfies that $n(0,0) = 0$ and $\frac{\partial n}{\partial x}(0,0) \neq 0$.*

Proof. By Lemma 2.7, we have that $n(0,0) = 0$. Suppose that $\frac{\partial n}{\partial x}(0,0) = 0$. Then, by differentiating both side of the equality in the assertion (ii) of Lemma 2.6 with respect to x, we have the following equality:

$$n_1(0,0)\frac{\partial^2 \Phi_1}{\partial x \partial y}(0,0) + \frac{\partial n_3}{\partial x}(0,0) = 0.$$

Since we have assumed $\frac{\partial n}{\partial x}(0,0) = 0$, we have that $\frac{\partial n_2}{\partial x}(0,0) = 0$. Thus and since Φ is a normalized Legendrian map-germ such that the origin of $\mathbb{R}^2$ is a singular point of Φ, we have that $\frac{\partial n_3}{\partial x}(0,0) \neq 0$. Thus, we have that $\frac{\partial^2 \Phi_1}{\partial x \partial y}(0,0) \neq 0$. Hence, by the condition (b) of the definition of normalized Legendrian maps, Lemma 2.7 and the equality (8), we have a contradiction. $\square$

Definition 3. Let $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a Legendrian map-germ and let ν_Φ be a unit normal vector field of Φ given in the definition of Legendrian map-germs. The C^∞ function-germ $LJ_\Phi : (\mathbb{R}^2, 0) \to \mathbb{R}$ defined by the following is called the *Legendrian-Jacobian* of Φ.

$$LJ_\Phi(x,y) = \det\left(\frac{\partial \Phi}{\partial x}(x,y), \frac{\partial \Phi}{\partial y}(x,y), \nu_\Phi(x,y)\right).$$

Note that if ν_Φ satisfies the conditions of unit normal vector field of Φ, then $-\nu_\Phi$ also satisfies them. Thus, the sign of $LJ_\Phi(x,y)$ depends on the particular choice of unit normal vector field ν_Φ. The Legendrian Jacobian of Φ is called also the *signed area density function* (for instance, see [9]). Although it seems reasonable to call LJ_Φ the area density function from the viewpoint of investigating the singular surface $\Phi(U)$ (U is a sufficiently small neighborhood of the origin of $\mathbb{R}^2$), it seems reasonable to call it the Legendrian Jacobian from the viewpoint of investigating the singular map-germ Φ. Hence, we call LJ_Φ the Legendrian Jacobian of Φ in this paper.

Let $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a normalized Legendrian map-germ and ν_Φ is a unit normal vector field of Φ.

$$\begin{aligned}\Phi(x,y) &= (\Phi_1(x,y), \Phi_2(x,y), y),\\ \nu_\Phi(x,y) &= n_1(x,y)\frac{\partial}{\partial X} + n_2(x,y)\frac{\partial}{\partial Y} + n_3(x,y)\frac{\partial}{\partial Z}.\end{aligned}$$

By Lemma 2.7, we may put

$$\widetilde{\nu}_\Phi(x,y) = \frac{\partial}{\partial X} + \frac{n_2(x,y)}{n_1(x,y)}\frac{\partial}{\partial Y} + \frac{n_3(x,y)}{n_1(x,y)}\frac{\partial}{\partial Z}.$$

Lemma 2.11. *The Legendrian Jacobian LJ_Φ is expressed as follows:*

$$LJ_\Phi(x,y) = \frac{\frac{\partial \Phi_2}{\partial x}(x,y)}{n_1(x,y)}.$$

Proof. Calculations show that

$$\frac{\partial \Phi}{\partial x}(x,y) \times \frac{\partial \Phi}{\partial y}(x,y) = \frac{\partial \Phi_2}{\partial x}(x,y)\widetilde{\nu}_\Phi(x,y)$$

where the cross in the center stands for the vector product. It follows $LJ_\Phi(x,y) = \frac{\partial \Phi_2}{\partial x}(x,y)/n_1(x,y)$. □

2.4. **Proof of Proposition 1.**

Proof of the assertion (i) of Proposition 1.

Put $\mathcal{I}(\varphi) = \Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y), y)$. Then, by Lemma 2.5, Φ is a Legendrian map-germ. Thus, in order to complete the proof of the assertion (i) of Proposition 1, it is sufficient to show the conditions (b), (c) of the definition of normalized Legendrian map-germs are satisfied.

Put $\varphi(x,y) = (n(x,y)p(x,y), p(x,y), y)$. Then, by the definition of map-germs of pedal unfolding type, we have that $n(0,0) = 0$ and $p(0,0) = 0$. It follows that $\frac{\partial \Phi_2}{\partial x}(0,0) = p(0,0) = 0$. Thus, the condition (b) is satisfied. By Lemma 2.4, the following L_Φ is a germ of Legendrian lift of Φ.

$$L_\Phi(x,y) = \left(\Phi(x,y), \frac{\widetilde{\nu}_\Phi(x,y)}{||\widetilde{\nu}_\Phi(x,y)||}\right),$$

where $\widetilde{\nu}_\Phi(x,y) = \frac{\partial}{\partial X} - n(x,y)\frac{\partial}{\partial Y} - \left(\frac{\partial \Phi_1}{\partial y}(x,y) - n(x,y)\frac{\partial \Phi_2}{\partial y}(x,y)\right)\frac{\partial}{\partial Z}$. Since $n(0,0) = 0$ and $\frac{\partial \Phi_1}{\partial y}(0,0) = \int_0^0 \frac{\partial np}{\partial y}(x,0)dx = 0$, we have

$$\frac{\widetilde{\nu}_\Phi(0,0)}{||\widetilde{\nu}_\Phi(0,0)||} = \frac{\partial}{\partial X}.$$

Thus the condition (c) is satisfied. □

Proof of the assertion (ii) of Proposition 1.

The assertion (ii) of Proposition 1 follows from Lemmas 2.8, 2.9 and 2.10. □

3. Proof of Theorem 1

Suppose that both $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ and $\mathcal{D} : \mathcal{S} \to \mathcal{W}$ are well-defined. Then, by the fundamental theorem of calculus, the following hold:

$$\begin{aligned}\mathcal{D} \circ \mathcal{I}(\varphi) &= \varphi \quad \text{for any } \varphi \in \mathcal{W},\\ \mathcal{I} \circ \mathcal{D}(\Phi) &= \Phi \quad \text{for any } \Phi \in \mathcal{S}.\end{aligned}$$

Thus, both $\mathcal{I}$ and $\mathcal{D}$ are bijective. Therefore, in order to complete the proof, it is sufficient to show that both $\mathcal{I}$ and $\mathcal{D}$ are well-defined.

Proof that $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ is well-defined.

Let $\varphi(x,y) = (n(x,y)p_\varphi(x,y), p_\varphi(x,y), y)$ be an element of $\mathcal{W}$. Put $\Phi = \mathcal{I}(\varphi)$. Then, Φ is a normalized Legendrian map-germ by Proposition 1 in §1. Let g be the Whitney umbrella of pedal unfolding type (2) defined in §1:

$$g(x,y) = (xp_g(x,y), p_g(x,y), y) = (x(x^2+y), -x^2-y, y).$$

Lemma 3.1. *There exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that h has the form $h(x,y) = (h_1(x,y), h_2(y))$ and $p_\varphi \circ h(x,y)$ is x^2+y or $-(x^2+y)$.*

Proof. Since φ is a Whitney umbrella of pedal unfolding type, we have the following:

$$Q(p_\varphi(x,y), y) \cong Q(\varphi) \cong Q(g) \cong Q(x^2, y).$$

Thus, we may put $p_\varphi(x,0) = a_2x^2 + o(x^2)$ $(a_2 \neq 0)$ by Proposition 2 in §2. By the Morse lemma with parameters (for instance, see [2]), there exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that h has the form $h(x,y) = (h_1(x,y), h_2(y))$ and $p_\varphi \circ h(x,y) = \pm(x^2+q(y))$ by a certain C^∞ function-germ $q : (\mathbb{R},0) \to (\mathbb{R},0)$. Since φ is $\mathcal{A}$-equivalent to g, by Corollary 1 in §2, $\pm(x^2+q(y))$ is $\mathcal{K}^T$-equivalent to p_g and thus $q : (\mathbb{R},0) \to (\mathbb{R},0)$ is a germ of C^∞ diffeomorphism. Hence, Lemma 3.1 follows. □

Lemma 3.2. *The normalized Legendrian map-germ Φ is a swallowtail.*

Proof. Put $G = \mathcal{I}(g)$. Then, G has the form (3) in §1 which is a normalized swallowtail. Since G is normalized, $\frac{\partial}{\partial x}$ is the null vector field for G defined in [4, 8], that is, $\frac{\partial G}{\partial x}(x,y) = 0$ holds for any (x,y) which is a singular point of G. Thus and since G is a swallowtail, the following three hold by corollary 2.5 of [8].

(i) $LJ_G(0,0) = \frac{\partial LJ_G}{\partial x}(0,0) = 0$,
(ii) $\frac{\partial^2 LJ_G}{\partial x^2}(0,0) \neq 0$.
(iii) $Q(LJ_G, \frac{\partial LJ_G}{\partial x}) \cong Q(x,y)$.

On the other hand, by Lemmas 2.11 and 3.1, there exist a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ and a C^∞ function-germ $\xi : (\mathbb{R}^2,0) \to \mathbb{R}$ such that h has the form $h(x,y) = (h_1(x,y), h_2(y))$, $\xi(0,0) \neq 0$ and the following hold:

$$LJ_\Phi \circ h(x,y) = \xi(x,y) LJ_G(x,y).$$

Hence and since $\frac{\partial}{\partial x}$ is the null vector field for Φ (this is because Φ is normalized), the following three hold for LJ_Φ.

(i) $LJ_\Phi(0,0) = \frac{\partial LJ_\Phi}{\partial x}(0,0) = 0$,
(ii) $\frac{\partial^2 LJ_\Phi}{\partial x^2}(0,0) \neq 0$,
(iii) $Q(LJ_\Phi, \frac{\partial LJ_\Phi}{\partial x}) \cong Q(x,y)$.

Hence, Φ is a swallowtail by corollary 2.5 of [8]. □

Proof that $\mathcal{D} : \mathcal{S} \to \mathcal{W}$ is well-defined.

Let Φ be an element of $\mathcal{S}$. Then, by Proposition 2 in §2, $\mathcal{D}(\Phi)$ is of pedal unfolding type.

Lemma 3.3. *For the Legendrian Jacobian LJ_Φ, the following three hold:*

(i) $LJ_\Phi(0,0) = \frac{\partial LJ_\Phi}{\partial x}(0,0) = 0$,
(ii) $\frac{\partial^2 LJ_\Phi}{\partial x^2}(0,0) \neq 0$.
(iii) $Q(LJ_\Phi, \frac{\partial LJ_\Phi}{\partial x}) \cong Q(x,y)$.

Proof. Since Φ is normalized, $\frac{\partial}{\partial x}$ is the null vector field. Thus and since Φ is a swallowtail, Lemma 3.3 follows from corollary 2.5 of [8].

Lemma 3.4. *For the Φ, the map-germ of pedal unfolding type $\mathcal{D}(\Phi)$ is a Whitney umbrella.*

Proof. Since $\mathcal{D}(\Phi)$ is of pedal unfolding type, there exists a C^∞ function-germ $n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ such that $\frac{\partial n}{\partial x}(0,0) \neq 0$ and $\frac{\partial \Phi_1}{\partial x}(x,y) = n(x,y)\frac{\partial \Phi_2}{\partial x}(x,y)$ where $\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y), y)$. Put $p_\varphi = \frac{\partial \Phi_2}{\partial x}$. Then, by Lemmas 2.11, 3.3 and the Morse lemma with parameters, there exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that h has the form $h(x,y) = (h_1(x,y), h_2(y))$ and $p_\varphi \circ h(x,y)$ is $x^2 + y$ or $-(x^2 + y)$. Therefore, by Corollary 1 in §2, $\mathcal{D}(\Phi)$ is $\mathcal{A}$-equivalent to g. □

4. Proof of Theorem 2

As same as Theorem 1, it is sufficient to show that both $\mathcal{I} : \mathcal{N} \to \mathcal{C}$ and $\mathcal{D} : \mathcal{C} \to \mathcal{N}$ are well-defined.

Proof that $\mathcal{I} : \mathcal{N} \to \mathcal{C}$ is well-defined.
Let $\varphi(x,y) = (n(x,y)p_\varphi(x,y), p_\varphi(x,y), y)$ be an element of $\mathcal{N}$. Put $\Phi = \mathcal{I}(\varphi)$. Then, since φ is of pedal unfolding type, Φ is a normalized Legendrian map-germ by Proposition 1 in §1. Let g be the non-singular map-germ of pedal unfolding type defined by $g(x,y) = (x^2, x, y)$.

Lemma 4.1. *There exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that h has the form $h(x,y) = (h_1(x,y), h_2(y))$ and $p_\varphi \circ h(x,y) = x$ holds.*

Proof. Since φ is non-singular and of pedal unfolding type, we have the following:
$$Q(p_\varphi(x,y), y) \cong Q(\varphi) \cong Q(g) \cong Q(x,y).$$
Thus, $(p_\varphi(x,y), y)$ is a germ of C^∞ diffeomorphism by Proposition 2 in §2. From the form of $(p_\varphi(x,y), y)$, its inverse map-germ $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ has the form $h(x,y) = (h_1(x,y), h_2(y))$. Since h is the inverse map-germ of $(p_\varphi(x,y), y)$, it follows that $p_\varphi \circ h(x,y) = x$. □

Lemma 4.2. *The normalized Legendrian map-germ Φ is a cuspidal edge.*

Proof. Since Φ is normalized, $\frac{\partial}{\partial x}$ is the null vector field for Φ. By Lemmas 2.11, 4.1, we have that $\frac{\partial LJ_\Phi}{\partial x}(0,0) \neq 0$. Thus, thus the null vector field $\frac{\partial}{\partial x}$ is transverse to $\{(x,y) \mid LJ_\Phi(x,y) = 0\}$ at $(0,0) \in \mathbb{R}^2$. Hence, Φ is a cuspidal edge by proposition 1.3 of [4]. □

Proof that $\mathcal{D} : \mathcal{C} \to \mathcal{N}$ is well-defined.
Let Φ be an element of $\mathcal{C}$. Then, by Proposition 1 in §1, $\mathcal{D}(\Phi)$ is of pedal unfolding type.

Lemma 4.3. *For the Legendrian Jacobian LJ_Φ, two properties $LJ_\Phi(0,0) = 0$ and $\frac{\partial LJ_\Phi}{\partial x}(0,0) \neq 0$ hold.*

Proof. Since $\frac{\partial}{\partial x}$ is the null vector field for Φ and Φ is a cuspidal edge, Lemma 4.3 follows from corollary 2.5 of [8].

Lemma 4.4. *For the* Φ*, the map-germ of pedal unfolding type* $\mathcal{D}(\Phi)$ *is non-singular and* $\mathcal{D}(\Phi)(0,0) = (0,0,0)$.

Proof. Since $\mathcal{D}(\Phi)$ is of pedal unfolding type, there exists a C^∞ function-germ $n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ such that $\frac{\partial n}{\partial x}(0,0) \neq 0$ and $\frac{\partial \Phi_1}{\partial x}(x,y) = n(x,y)\frac{\partial \Phi_2}{\partial x}(x,y)$ where $\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y), y)$. Put $p_\varphi = \frac{\partial \Phi_2}{\partial x}$. Then, by Lemmas 2.11 and 4.3, the map-germ $(x,y) \mapsto (p_\varphi(x,y), y)$ is a germ of C^∞ diffeomorphism. Thus, $\mathcal{D}(\Phi)$ is non-singular. $\square$

Acknowledgement. The author would like to thank Kentaro Saji for valuable comments.

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